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American Journal of Engineering & Natural Sciences (AJENS)
Volume 1, Issue 1, August 2016

On The Generation Of Physically Acceptable Einstein Equation Solutions For Static Perfect Fluid Spheres

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***Abstract*— In [1] some techniques for generating new solutions from known solutions have been presented. In this paper, applying one of these techniques to the Schwarzschild constant density solution we have generated a 1-parameter family of new solutions.**

Keywords— Perfect fluid, Static Perfect fluid, Centre of symmetry, Decreasing function, Regular boundary surface, Constant density solution.

I. Introduction

Exact analytic solutions of Einstein's equations are difficult because of the high nonlinearity of the equations. In this paper we consider static spherically symmetric perfect fluid solutions. Even for this simplest case only few solutions (about 16) are known which fulfill all the requirements for physical acceptability of the solutions. For physical acceptability static perfect fluid sphere solutions are required to satisfy the following properties:

(1) Both p(r) and $\rho(r)$ are required to be positive definite at the centre of symmetry r = 0.

(2) p(r) is required to vanish at some finite radius R > 0.

(3) Both p(r) and $\rho(r)$ should be decreasing functions of r.

(4) It is required that $\frac{dp}{d\rho}\langle 1$.

For a static spherically symmetric system metric of space-time in curvature coordinates has the following form

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2 d\Omega^2 \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. For the metric (1), (r, r) and (θ,θ) components of Einstein's equations give

$$m'(r) + \frac{2r^2N'' - 3rN' - 3N}{r(rN' + N)}m(r) = \frac{r^2N'' - rN'}{rN' + N} \quad (2)$$

$$\text{or, } f'(r) + \frac{2rN''}{rN' + N}f(r) = \frac{rN'' - N'}{r^2(rN' + N)} \quad (3)$$

where $f(r) = \frac{m(r)}{r^3}$.

If we put $N(r) = e^{\varphi(r)}$, then equation (3) can be expressed in terms of $\varphi(r)$ as follows

$$f'(r) + \frac{2r(\varphi'' + \varphi'^2)}{r\varphi' + 1}f(r) = \frac{r\varphi'' + r\varphi'^2 - \varphi'}{r\varphi' + 1} \quad (4)$$

(t, t) component of Einstein's equation gives

$m'(r) = 4\pi r^2\rho$

$$\text{or, } \rho(r) = \frac{m'(r)}{4\pi r^2} \geq 0 \quad (5)$$

Also the (r, r) component of Einstein's equation can be cast into the form

$$p(r) = \frac{r\varphi'(r-2m) - m}{4\pi r^3} \geq 0 \qquad (6)$$

The inequalities (5) and (6) require that $\varphi(0)$ must be a finite constant such that $\varphi'(0) = 0$ and $\varphi''(0) > 0$. Also the necessary and sufficient condition that the solution have a regular boundary surface with Schwarzschild vacuum exterior at r = R > 0 is given by p(R) = 0. Putting m(R) = M it follows from (6) that

$$\varphi'(R) = \frac{M}{R(R-2M)}$$

Now the conditions on $\varphi(r)$ for the solutions to be regular at r = 0 require that N(0) should be a finite constant such that $N'(0) = 0$ and $N''(0) > 0$.

II. Generation of Solutions Which Are Regular At r = 0

Equation (3) is a first order linear differential equation in f(r), (and hence m(r)), which can be solved if N(r) is chosen. If N(r) is chosen in such a way that it satisfies the conditions mentioned in Section-1 the solutions obtained will be regular at r = 0. [However this does not ensure the fulfillment of the other requirements for the solutions to be physically acceptable].

In the following we demonstrate this by constructing some exact known solutions.

Let N(r) = $(ar^2+1)^{\frac{n}{2}}$, where n is a positive integer. Then N(0) = 1, $N'(0) = 0$, $N''(0) > 0$.

Then we obtain the following solutions:

(1) Tolman IV solution for n = 1, [Ref. [2])

$$ds^2 = -(1+ar^2)dt^2 + \frac{1+2ar^2}{(1-\frac{r^2}{b^2})(1+ar^2)}dr^2 + r^2 d\Omega^2$$

(2) Adler's solution for n = 2, (Ref. [3])

$$ds^2 = -(1+ar^2)^2 dt^2 + \frac{dr^2}{1+\frac{cr^2}{(1+3ar^2)^{\frac{2}{3}}}} + r^2 d\Omega^2$$

(3) H. Heintzmann IIa solution for n = 3, (Ref. [4])

$$ds^2 = -(1+ar^2)^3 dt^2 + \frac{dr^2}{1-\frac{3ar^2}{2}\cdot\frac{c(1+4ar^2)^{-\frac{1}{2}}}{1+ar^2}} + r^2 d\Omega^2$$

(4) Durgapal IV solution for n = 4, (Ref. [5])

$$ds^2 = -(1+ar^2)^4 dt^2 +$$

$$\{\frac{7-10ar^2-a^2r^4}{7(1+ar^2)^2} + \frac{kar^2}{(1+ar^2)^2(1+5ar^2)^{\frac{2}{5}}}\}dr^2 + r^2 d\Omega^2$$

(5) Durgapal V solution for n = 5, (Ref. [5])

$$ds^2 = -(1+ar^2)^5 dt^2 +$$

$$\left\{\frac{1-\frac{ar^2(309+54ar^2+8a^2r^4)}{112}+\frac{kar^2}{\sqrt[3]{1+6ar^2}}}{(1+ar^2)^3}\right\}dr^2 + r^2 d\Omega^2$$

(6) Buchdahl I solution for n = $\frac{3}{2}$, [for $b = \infty$], (Ref. [6])

$$ds^2 = -(1+ar^2)^{\frac{3}{2}} dt^2 + \frac{1+ar^2}{1-\frac{a}{2}r^2}dr^2 + r^2 d\Omega^2$$

III. New Solution From Known Solution

In [1] some techniques for generating new solutions from known solution have been presented. In the following we briefly describe these techniques.

Equation (2) can be rewritten as follows,

$$G'(r) + \frac{2(r^2N'' - rN' - N)}{r(rN' + N)}G + \frac{2N}{r(rN'+N)} = 0 \qquad (7)$$

where $G(r) = 1 - \frac{2m}{r}$.

Equation (7) can be regrouped as

$$(2r^2G)N'' + (r^2G' - 2rG)N' + (rG' - 2G + 2)N = 0 \qquad (8)$$

Suppose $(N_0(r), G_0(r))$ is known solution of (7) i.e. we suppose that the equation

$$G_0' + \frac{2(r^2 N_0'' - rN_0' - N_0)}{r(rN_0' + N_0)} G_0 + \frac{2N_0}{r(rN_0' + N_0)} = 0 \qquad (9)$$

is satisfied. Then it can be verified that, (i) $(N_0, G_0(r) + k\Delta_0(r))$, where k is a constant and

$$\Delta_0(r) = \frac{1}{[(rN_0)']^2} \exp \int \frac{4N_0'}{rN_0' + N_0} dr \qquad (10)$$

is a solution of (7).

(ii) $(N_0 Z_0, G_0)$, where

$$Z_0 = c_1 + c_2 \int \frac{rdr}{N_0^2 \sqrt{G_0}}$$

is a solution of (7).

The results can be viewed as the transformations

$$T_1 : (N_0, G_0) \rightarrow (N_0, G_0(r) + k\Delta_0(r))$$

$$T_2 : (N_0, G_0) \rightarrow (N_0 Z_0, G_0), G_0)$$

The composite transformations T_3 , T_4 defined by

$$T_3(N_0, G_0) = T_2(T_1(N_0, G_0))$$

$$T_4(N_0, G_0) = T_1(T_2(N_0, G_0))$$

also provide new solutions. Thus the transformations T_1 , T_2 , T_3 , T_4 defined when applied to a known solution provide new solutions.

IV. New Solutions

In this section we have generated a class of new solutions using the technique described in the previous section. For this we apply transformation T_1 to the Schwarzschild constant density solution [7] given by

$$(N_0, G_0) = (l - \sqrt{1 - ar^2}, 1 - ar^2)$$

where a, l are arbitrary constants. Then from (10) we obtain

$$\Delta_0(r) = \frac{r^2}{\{\frac{d}{dx}(lr - r\sqrt{1 - ar^2})\}^2}$$

$$\times \exp \int \frac{-4ardr}{2(1 - ar^2) - l\sqrt{1 - ar^2} - 1}$$

$$= \frac{r^2(1 - ar^2)}{(2ar^2 - 1 + l\sqrt{1 - ar^2})^2}$$

$$\times \exp \int \frac{-4ardr}{2(1 - ar^2) - l\sqrt{1 - ar^2} - 1}$$

$$= \frac{r^2 x^2}{(1 - 2x^2 + lx)^2} \exp \int \frac{4xdx}{2x^2 - lx - 1}$$

where $x^2 = 1 - ar^2$

But $\int \frac{4xdx}{2x^2 - lx - 1} =$

$$\log \left\{ (2x^2 - lx - 1) \left(\frac{4x - l - \sqrt{l^2 + 8}}{4x - l + \sqrt{l^2 + 8}} \right)^{\frac{l}{\sqrt{(l^2 + 8)}}} \right\}$$

So we get

$$\Delta_0(r) = \frac{r^2 x^2}{2x^2 - lx - 1} \times \left(\frac{4x - l - \sqrt{l^2 + 8}}{4x - l + \sqrt{l^2 + 8}} \right)^{\frac{l}{\sqrt{(l^2 + 8)}}}$$

Conclusion

Starting from the known solution $(N_0, G_0) = (l - \sqrt{1 - ar^2}, 1 - ar^2)$ we have generated a 1-parameter family of new solutions using the transformation T_1. For each value of the parameter l we get a new solution. For $l = 0$ we get Tolman solution IV. Another class of solutions can be generated by applying the transformation T_2 to the Schwarzschild constant density solution.

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