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A ZERO - STABLE HYBRID LINEAR MULTISTEP METHOD FOR THE NUMERICAL SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, a new Zero – stable continuous hybrid linear multistep method is proposed for the numerical solution of initial value problems of first order ordinary differential equations. The procedure for derivation of the numerical method entails obtaining the main method and additional schemes from the same continuous scheme derived via interpolation and collocation procedure. The implementation of the method is by applying the main scheme together with the additional schemes in a block form as simultaneous integrators over non – overlapping intervals. The method was found to be consistent, zero – stable, convergent and accurate.

Mathematics Subject Classification: 65L80

Key words: Hybrid, linear multi step, numerical, block method, interpolation and collocation.

1.0 Introduction

Our focus in this paper is the construction of numerical method for the solution of initial value problems of first order Ordinary differential equations of the general form:

$$y' = f(x, y), \qquad y(x_0) = y_0, x \in [x_{0,} b], \qquad (1)$$

Where y' indicates the derivative of dependent variable y with respect to x and function f satisfies the Lipschitz condition of the existence and uniqueness of solution to the ordinary differential equation. Wide varieties of natural phenomena in physics, engineering, biology and social sciences, are modeled by ordinary differential equations of the general form (1). Even though, the conventional method of solving (1) is analytical methods which include: separation of variables, integrating factor, transformation method and so on, however, not all ordinary differential equations can be solved analytically. Hence, the earlier researchers proposed numerical method of solution to initial value problems of ordinary differential equations. Consequently, the solution of (1) has been discussed extensively by various researchers which include: [1] – [13], to mention just a few.

Collocation methods have its origin dated back to as far as 1965, when [14] introduced the standard collocation method with some selected points for the numerical integration of the ordinary differential equations. It should be noted that the earlier researchers that proposed collocation methods for solving ODEs developed discrete schemes; it was [15] that showed that traditional multistep methods including the hybrid ones can be made continuous through the ideal of multistep collocation scheme against the discrete schemes since global error estimates can be attained in addition to better approximation at all interior points. Thus, the introduction of continuous collocation methods has bridged the gap between the discrete collocation method and the conventional multistep method. Proposition of hybrid continuous collocation methods is now commonly in vogue for the numerical solution of ordinary differential equations apparently because they are efficient, accurate and adequate. The works of [16] and [17] attest to this.

In this paper, we present a continuous two – step hybrid block method employing multistep collocation approach which produces a class of four schemes of order 5 for the numerical integration of initial value problems of first order ordinary differential equations.

2. Derivation of the Method.

We seek a K – step multistep collocation method of the form:

$$\sum_{j=1}^{2}\alpha_j y_{n+j} = h\left[\sum_{j=0}^{2}\beta_j f_{n+j} + \beta_v f_{n+v}\right], \quad (2)$$

Where α_j and β_j are coefficients and $v = \left\{\frac{4}{3}, \frac{5}{3}\right\}$ are hybrid points. We assume an approximate solution to equation (1) to be a continuous solution of the form:

$$y(x) = \sum_{j=0}^{p+q-1}\alpha_j x^j \quad (3)$$

Such that $x \in [x_0, b]$, where a_j are unknown coefficients of the polynomial basis function of degree $p+q-1$, where the number of interpolation points p and the number of collocation points q are respectively chosen to satisfy $1 \le p \le k$ and $q \succ 0$. Note that the step number of the method is represented by $k \succ 1$.

We construct a k – step continuous hybrid multistep method with $x^j,\ j = 0,1,\ldots,5,\ p = 1, q = 5, k = 2$ by imposing the above condition, we have:

$$\sum_{j=0}^{5} j a_j x_{n+1}^{j-1} = f_{n+i}, i = \left\{0,1,\frac{4}{3},\frac{5}{3},2\right\}, \quad (4)$$

$$\sum_{j=0}^{5} j a_j x_{n+1}^{j-1} = y_{n+i}, i = 1 \quad (5)$$

Where n in (4) in (5) above is the grid index.

From equations (4) and (5) we obtain a system of $p+q$ equations which are solved to obtain the coefficients a_j 's using Gaussian elimination method. By putting the values of these coefficients a_j 's so obtained into equation (3), we obtain the two – step continuous hybrid method. On evaluating the continuous scheme at points $x = \{x_n, x_{n+1}, x_{n+4/3}, x_{n+5/3}, 2\}$, we obtain the following four discrete schemes:

$$y_{n+2} - y_{n+1} = h\left[-\tfrac{1}{1200} f_n + \tfrac{17}{120} f_{n+1} + \tfrac{27}{80} f_{n+4/3} + \tfrac{81}{200} f_{n+5/3} + \tfrac{7}{65} f_{n+2}\right] \quad (6)$$

$$y_{n+5/3} - y_{n+1} = h\left[-\tfrac{1}{4050} f_n + \tfrac{47}{405} f_{n+1} + \tfrac{13}{30} f_{n+4/3} + \tfrac{3}{25} f_{n+5/3} - \tfrac{1}{405} f_{n+2}\right] \quad (7)$$

$$y_{n+4/3} - y_{n+1} = h\left[-\tfrac{19}{32400} f_n + \tfrac{443}{3240} f_{n+1} + \tfrac{19}{80} f_{n+4/3} - \tfrac{29}{600} f_{n+5/3} + \tfrac{13}{1620} f_{n+2}\right] \quad (8)$$

$$y_n - y_{n+1} = h\left[-\tfrac{329}{1200} f_n - \tfrac{287}{120} f_{n+1} + \tfrac{243}{80} f_{n+4/3} - \tfrac{351}{200} f_{n+5/3} + \tfrac{23}{60} f_{n+2}\right] \quad (9)$$

These are the discrete schemes that will be used as simultaneous integrator of test problems of initial value first order ordinary differential equations.

3. ANALYSIS OF THE BASIC PROPERTIES OF THE METHOD.

3.1 Order of accuracy and Error Constant.

The local truncation error associated with K – step linear multistep method (2), in line with [18], is obtained from the linear difference operator:

$$L[y(x), h] = \sum_{j=0}^{k} \{\alpha_j y(x_{n+j}) - h\beta_j y(x_{n+j})\} \quad (10)$$

Equation (10) can be expanded as a Taylor's series about the point x if $y(x)$ is sufficiently differentiable to obtain the expression:

$$L[y(x), h] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \ldots + C_2 h^q y^q(x_n) + \ldots, \quad (11)$$

where $C_{q,}$ $q = 0,1,\ldots,$are the constant coefficients given as:

$$C_0 = \sum_{j=0}^{k} \alpha_j, \quad (12)$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j, \quad (13)$$

And

$$C_q = \tfrac{1}{q!}\left[\sum_{j=0}^{k} j\alpha_j - q(q-1)\left(\sum_{j=0}^{k} j^{q-1}\beta_j + \sum_{j=0}^{k} v^{q-1}\beta_{vj}\right)\right], \quad (14)$$

In line with [18], according to (11), we say that the hybrid k – step, linear multistep method (2) has order p if $C_0 = C_1 = \ldots = C_{p-1} = C_p$ and $C_{p+1} \neq 0$. Thus C_{p+1} is the error constant of the method. From our calculation, subjecting our schemes to equations (11)-(14), it is established that our hybrid linear multistep scheme have high order of accuracy and relatively small error constant as tabulated below:

Table 1.

Order and Error Constant of the schemes.

Equation number	Order of accuracy	Error Constants
6	5	-1.264108 $\times 10^{-3}$
7	5	0.612002 $\times 10^{-3}$
8	5	0.526255 $\times 10^{-2}$
9	5	-0.162842 $\times 10^{-1}$

3.2 Consistency

The linear multistep method (2) is said to be consistent if:

(i) The order $p \geq 1$,

(ii) $\sum_{j=0}^{k} \alpha_j = 0,$

(iii) $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j,$

(iv) $\rho(1) = 0, \rho'(1) = \sigma(1),$

Where ρ and σ are the first and second characteristic polynomials of equation (2), the general form of our method. By applying these aforelisted definitions to our schemes (6-9), they were found to be consistent.

3.3 Zero Stability of the method.

According to [18], a linear multistep method of the form (2) is said to be Zero stable if no roots of the first characteristic polynomial $\rho(r)$ has modulus greater than one, and if every root of modulus one is simple. In the same way, by applying this definition to our schemes (6-9), they were found to be Zero stable.

4. Implementation of the method.

The mode of implementation of our method is by combining the hybrid linear multistep schemes (6-9) as simultaneous integrator for the initial value problems of first order ordinary differential equations without requiring starting values and predictors. In doing this, we proceed by explicitly obtaining initial conditions at $x_{n+2}, n = 0,2,\ldots,N-2,$ using the computed values: $y(x_{n+2}) = y_{n+2}$ over sub intervals $[x_0, x_2],\ldots,[x_{N-2}, x_N]$. specifically, we use equations (6-9) setting n = 0, $\mu = 0$.we obtain simultaneously $\left(y_1, y_{4/3}, y_{5/3}, y_2\right)^T$, over the sub interval ,$[x_0, x_2]$, as y_0 is known from the initial value problem (1). In the same way, by setting n = 2, $\mu = 1$, we obtain simultaneously $\left(y_3, y_{10/3}, y_{11/3}, y_4\right)^T$, over the sub interval ,$[x_2, x_4]$ as y_2 is known from the previous block, T being the transpose. This is process is continued until it is convenient to stop the iterations. Hence, the sub – interval do not overlap, thus the solution obtained herefrom are more accurate than those obtain in the conventional fashion. However, for linear problems, equation (1) is solved from the start by Gaussian elimination method using partial pivoting, while for nonlinear problems, modified Newton - Raphson method can be used.

4.1 Numerical Results.

The numerical schemes (6-9) is coded with the aid of MATLAB software and implemented on test problems to test the suitability and performance of the method. The results are presented in the tabular form.

Test Problem 1.

We consider a stiff initial value problem:

$y' = -10(y - x^3) + 3x^2 \qquad y(0) = 1$ with h = 0.1

Whose exact solution is $y(x) = x^3 + e^{-10x}$

The results are as shown in table 2 below.

Table 2.

Results for Problem 1.

x	Exact Solution	Numerical Solution	Error.
0	1	1	0.0
0.1	0.36788044	0.36788003	4.1×10^{-06}
0.2	0.60653078	0.60653046	3.2×10^{-06}
0.3	0.77800800	0.77800786	1.4×10^{-06}
0.4	0.88249690	0.88249648	4.2×10^{-06}
0.5	0.93941306	0.93941273	3.3×10^{-06}
0.6	0.96923323	0.96923311	1.2×10^{-06}
0.7	0.98449644	0.98449572	7.2×10^{-06}
0.8	0.99221794	0.99221770	2.4×10^{-06}
0.9	0.99610137	0.99610125	1.2×10^{-06}

Test Problem 2.

We consider the initial value problem given by:

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1, \qquad 0 \le x \prec \pi/2,\ y(\pi/4) = 3\sqrt{2},\ with\ h = \pi/100$$

Whose exact solution is given by $y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$

Our results were compared with that of [6]. The results are as shown in the table 3 below.

Table 3.

Results for Problem 2.

x	Exact Solution	Numerical Solution	Error	Error in [6]
0	6.50000000	6.50000000	0.0	0.0
$\pi/100$	6.46636803	6.46636777	2.6×10^{-06}	9.1×10^{-05}
$\pi/10$	5.96366817	5.96366755	6.2×10^{-06}	1.1×10^{-04}
$19\pi/100$	5.07524630	5.07524599	3.1×10^{-06}	1.2×10^{-04}
$\pi/4$	4.24264068	4.24264024	4.4×10^{-06}	1.3×10^{-04}
$7\pi/25$	3.75117535	3.75117510	2.5×10^{-06}	1.4×10^{-04}
$37\pi/100$	1.99821382	1.99821366	1.6×10^{-06}	1.5×10^{-04}
$23\pi/50$	-0.05408423	-0.05408331	9.2×10^{-06}	1.5×10^{-04}
$47\pi/100$	-0.29058330	-0.29058309	2.1×10^{-06}	1.5×10^{-04}
$12\pi/25$	-0.52734539	-0.52734523	1.6×10^{-06}	1.6×10^{-04}

5. Conclusion.

The method proposed in this paper as can be seen from the results of its numerical implementation when adopted to solve initial value problems of fist order ordinary differential equations are consistent and convergent and can compete favourably with existing methods.

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